

Knot Invariants and New Quantum Field Theory

Sze Kui Ng

Department of Mathematics, Hong Kong Baptist University, Hong Kong

Abstract

We propose a new gauge theory of quantum electrodynamics (QED) and quantum chromodynamics (QCD) from which we derive knot invariants such as the Jones polynomial. Our approach is inspired by the work of Witten who derived knot invariants from quantum field theory based on the Chern-Simon Lagrangian. From our approach we can derive new knot invariants which extend the Jones polynomial and give a complete classification of knots.

1 Introduction

In 1989 Witten derived knot invariants such as the Jones polynomial from quantum field theory based on the Chern-Simon Lagrangian [1]. Inspired by Witten's work in this paper we shall derive knot invariants from a new gauge theory of quantum electrodynamics (QED) and quantum chromodynamics (QCD) [2] [3]. Comparing to Witten's work our approach will be simpler and more rigorous. In our approach we shall first derive the Jones polynomial and then by a quantum group structure we derive new knot invariants which extend the Jones polynomial.

This paper is organized as follows. In section 2 and section 3 we give a brief description of a new gauge theory of QED and QCD. In this paper we shall consider a QCD with a $SU(2)$ gauge symmetry. With this new quantum field theory we introduce the partition function and the correlation of Wilson loop which will be a knot invariant of the trivial knot. This correlation of Wilson loop will later be generalized to be knot invariants of nontrivial knots. To investigate the properties of these partition function and correlations in section 4 we derive a chiral symmetry from the gauge transformation of this new quantum field theory. From this chiral symmetry in section 5, section 6 we derive a conformal field theory which contains topics such as the affine Kac-Moody algebra and the Knizhnik-Zamolodchikov equation. This KZ equation is an equation of correlations from which in section 7 we can derive the skein relation of the Jones polynomial. A main point of our theory on the KZ equation is that we can derive two KZ equations which are dual to each other. From these two

KZ equations we derive a quantum group structure for the W matrices from which a Wilson loop is formed. Then from the correlation of these W matrices in section 8 we derive new knot invariants which extend the Jones polynomial and gives a classification of knots.

2 New gauge theory of QED

To begin our derivation of knot invariants let us first describe a new quantum field theory. Similar to the Wiener measure for the Brownian motion which is constructed from the integral $\int_{t_0}^{t_1} \left(\frac{dx}{dt}\right)^2 dt$ we construct a measure for QED from the following energy integral:

$$-\frac{1}{2} \int_{s_0}^{s_1} \left[\frac{1}{2} \left(\frac{dA_1}{ds} - \frac{dA_2}{ds} \right)^2 + \sum_{i=1}^2 \left(\frac{dz}{ds} - ieA_i z \right)^* \left(\frac{dz}{ds} - ieA_i z \right) \right] ds \quad (1)$$

where s denotes the proper time in relativity; e denotes the electric charge and the complex variable z , real variables A_1, A_2 represent one electron and two photons respectively. By extending ds to $(ih + \beta)ds$ we get a quantum theory of QED where $h > 0$ denotes the Planck constant and $\beta > 0$ is a constant related to absolute temperature. The integral (1) has the following gauge symmetry:

$$z'(s) = z(s)e^{iea(s)}, \quad A'_i(s) = A_i(s) + \frac{da}{ds} \quad i = 1, 2 \quad (2)$$

where $a(s)$ is a real valued function.

Let us briefly give some physical motivations for the above QED theory. A main feature of (1) is that it is not formulated with the four-dimensional space-time. As most of the theories in physics are formulated with the space-time that we want to explain the reason of our formulation. We know that with the concept of space-time we have a convenient way to understand physical phenomena and to formulate theories of physics such as the Newton equation, the Schrodinger equation for quantum mechanics, e.t.c. to describe these phenomena. However we also know that there are fundamental difficulties related to the space-time such as the ultraviolet divergences in quantum field theory [4] [5]. To solve these difficulties let us reexamine the concept of space-time. We propose that the space-time is a statistical concept which is not as basic as the proper time in relativity [2]. Because a statistical theory is usually a convenient but incomplete description of a more basic theory this means that some difficulties may appear if we formulate a physical theory with the space-time. This also means that a way to formulate a basic theory of physics is to formulate it not with the space-time but with the proper time only as the parameter for evolution. This is a reason that we use (1) to formulate a QED theory. In this formulation we regard the proper time as an independent parameter which is not in terms of the space-time. From (1) we may obtain the

conventional results in terms of space-time by introducing the space-time as a statistical method [2] [3].

Let us explain in more detail how the space-time come out as a statistics. For statistical purpose when many electrons (or many photons) present we introduce space-time (t, \mathbf{x}) as a statistical method to write $s - s_0$ in the form $\omega(s - s_0) = \omega(t - t_0) - \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_0)$ and to write ds^2 in the form

$$ds^2 = dt^2 - d\mathbf{x}^2 \quad (3)$$

We notice that for a given ds there may have many dt and $d\mathbf{x}$ which correspond to many electrons (or photons) such that (3) holds. In this way the space-time (t, \mathbf{x}) is introduced as a statistics. By these two relations we can derive statistical formulas for many electrons (or photons) from formulas obtained from (1). In this way we may obtain the Dirac equation as a statistical equation for electrons and the Maxwell equation as a statistical equation for photons. From this statistical method and the form of (1) we may show that why electrons follow Fermi-Dirac statistics and photons follow Bose-Einstein statistics [2] [3]. We notice that the relation (3) is the famous Lorentz metric. Here our understanding of the Lorentz metric is that it is a statistical formula where the proper time s is more fundamental than the space-time (t, \mathbf{x}) in the sense that we first have the proper time and then the space-time is introduced via the Lorentz metric only for the purpose of statistics. This reverses the order of the appearance of the proper time and the space-time in the history of relativity in which we first have the concept of space-time and then we have the concept of proper time which is introduced via the Lorentz metric. Once we understand that the space-time is a statistical concept from (1) we can give a solution to the debate about quantum mechanics between Bohr and Einstein [2] [4]. In this debate Bohr insisted that with probability interpretation quantum mechanics is very successful. On the other hand Einstein insisted that quantum mechanics is incomplete because of probability interpretation. He believed that God does not play dice. Here we may solve this debate by constructing the above QED theory which is a quantum theory and unlike quantum mechanics it is deterministic since it is not formulated with the space-time. Here we understand that quantum mechanics needs probability interpretation because it is formulated with the space-time which is a statistical concept.

From (1) we may derive QED effects including the abnormal magnetic moment and the Lamb shift [3]. Because the proper time s is an one dimensional parameter we have that this new QED theory is free of the difficulty of ultraviolet divergences and is a renormalizable theory. This is an original motivation that we set up (1) as a QED theory.

3 $SU(2)$ Gauge Symmetry

We can generalize the above QED theory with $U(1)$ gauge symmetry to QCD type theories with nonabelian gauge symmetry. As an illustration let us consider $SU(2)$ gauge symmetry. Similar to (1) we consider an energy integral L of the following form:

$$-\frac{1}{2} \int_{s_0}^{s_1} \left[\frac{1}{2} \text{tr}(D_1 A_2 - D_2 A_1)^* (D_1 A_2 - D_2 A_1) + (D_1 Z)^* (D_1 Z) + (D_2 Z)^* (D_2 Z) \right] ds \quad (4)$$

where $Z = (z_1, z_2)^T$ is a two dimensional complex vector; $A_j = \sum_{k=1}^3 A_j^k t^k$ ($j = 1, 2$) where A_j^k denotes a real component of a gauge field A^k ; t^k denotes a generator of $SU(2)$; and $D_j = \frac{d}{ds} - igA_j$ ($j = 1, 2$) where g denotes the charge of interaction. From (4) we can develop a QCD type model as similar to that for the QED model. We have that (4) is invariant under the following gauge transformation:

$$\begin{aligned} Z'(s) &= U(a(s))Z(s) \\ A'_j(s) &= U(a(s))A_j(s)U^{-1}(a(s)) + U(a(s))\frac{dU^{-1}(a(s))}{ds}, j = 1, 2 \end{aligned} \quad (5)$$

where $U(a(s)) = e^{-a(s)}$ and $a(s) = \sum_k a^k(s)t^k$.

Since (4) is not formulated with the space-time, as analogous to the approach of Witten on knot invariants [1] the following partition function will be shown to be a topological invariant for knots:

$$\langle \text{Tr} W_R \rangle := \int DA_1 DA_2 DZ e^L \text{Tr} W_R(C) \quad (6)$$

where

$$W_R(C) := W(r_0, r_1) := P e^{\int_C A_i dx^i} \quad (7)$$

which may be called a Wilson loop as analogous to the usual Wilson loop [1] where C denotes a closed curve of the following form

$$C(s) = (x^1(r(s)), x^2(r(s))), s_0 \leq s \leq s_1 \quad (8)$$

where r is a real valued function such that $r_0 := r(s_0) = r(s_1) := r_1$. This closed curve C is in a two dimensional phase plane (x^1, x^2) which is dual to (A_1, A_2) . We let this closed curve C represents the projection of a knot in this two dimensional space. As usual the notation P in the definition of $W_R(C)$ denotes a path-ordered product and R denotes a representation of $SU(2)$ [6] [7]. We remark that we also extend the definition of $W_R(C)$ to the case that C is not a closed curve with $r_0 \neq r_1$.

We shall show that (6) is a topological invariant for a trivial knot. This means that in (6) the closed curve C represents a trivial knot. We shall extend (6) to let C represent nontrivial knot. Our aim is to compute the above knot

invariant and to derive the Jones polynomials which are knot invariants for $SU(2)$ [1] by using Chern-Simon quantum field theory. Then we derive new knot invariants which extend the Jones polynomial.

4 Chiral Symmetry

For a given curve $C(s) = (x^1(r(s)), x^2(r(s)))$, $s_0 \leq s \leq s_1$ which may not be a closed curve we define $W(r_0, r_1)$ by (7) where $r_0 = r(s_0)$ and $r_1 = r(s_1)$. Then under a gauge transformation (5) where we let a be a function of the form $a(s) = \omega(r(s))$ for some analytic function ω we have the following chiral symmetry:

$$W(r_0, r_1) \mapsto U(\omega(r_1))W(r_0, r_1)U^{-1}(\omega(r_0)) \quad (9)$$

This chiral symmetry is analogous to the chiral symmetry of the usual non-abelian gauge theory where U denotes an element of $SU(2)$ [6]. We may extend (10) by extending r to complex variable z to have the following chiral symmetry:

$$W(z_0, z_1) \mapsto U(\omega(z_1))W(z_0, z_1)U^{-1}(\omega(z_0)) \quad (10)$$

This analytic continuation corresponds to the complex transformation $s \mapsto (ih + \beta)s$ for describing quantum physics.

From this chiral symmetry we have the following formulas for the variations $\delta_\omega W$ and $\delta_{\omega'} W$ with respect to the chiral symmetry:

$$\delta_\omega W(z, z') = W(z, z')\omega(z) \quad (11)$$

and

$$\delta_{\omega'} W(z, z') = -\omega'(z')W(z, z') \quad (12)$$

where z and z' are independent variables and $\omega'(z') = \omega(z)$ when $z' = z$. In (11) the variation is with respect to the z variable while in (12) the variation is with respect to the z' variable.

5 Affine Kac-Moody Algebra

Let us define

$$J(z) := -kW^{-1}(z, z')\partial_z W(z, z') \quad (13)$$

where $k > 0$ is a constant. As analogous to the WZW model [9] [11] J is a generator of the chiral symmetry for (11). In the quantum case J satisfies some conditions to be specified below by adjusting the Planck constant \hbar once the constant k is fixed. Let us consider the following correlation

$$\langle W_R A(z) \rangle := \int DA_1 DA_2 DZ e^L W_R(C) A(z) \quad (14)$$

By taking a gauge transformation on this correlation and by the gauge invariance of (4) we can derive a Ward identity from which we have the following relation:

$$\delta_\omega A(z) = \frac{-1}{2\pi i} \oint_z dw \omega(w) J(w) A(z) \quad (15)$$

where $\delta_\omega A$ denotes the variation of the field A with respect to the chiral symmetry and the closed line integral \oint is with center z and we let the generator J be given by (13). We remark that our approach here is analogous to the WZW model in conformal field theory [11].

From (10) and (13) we have that the variation $\delta_\omega J$ of the generator J of the chiral symmetry is given by [9] [11]:

$$\delta_\omega J = [J, \omega] - k \partial_z \omega \quad (16)$$

From (15) and (16) we have that J satisfies the following relation of current algebra [9]:

$$J^a(w) J^b(z) = \frac{k \delta_{ab}}{(w-z)^2} + \sum_c i f_{abc} \frac{J^c(z)}{(w-z)} \quad (17)$$

where we write

$$J(z) = \sum_a J^a(z) t^a = \sum_a \sum_{n=-\infty}^{\infty} J_n^a z^{-n-1} t^a \quad (18)$$

Then from (17) we have that J_n^a satisfy the following affine Kac-Moody algebra[9]:

$$[J_m^a, J_n^b] = i f_{abc} J_{m+n}^c + k m \delta_{ab} \delta_{m+n,0} \quad (19)$$

where the constant k is called the central extension or the level of the Kac-Moody algebra.

Let us consider another generator of the chiral symmetry for (12) given by

$$J'(z') = k \partial_{z'} W(z, z') W^{-1}(z, z') \quad (20)$$

Similar to J by the following correlation:

$$\langle A(z') W_R \rangle := \int DA_1 DA_2 DZ A(z') W_R(C) e^L \quad (21)$$

we have the following formula for J' :

$$\delta_{\omega'} A(z') = \frac{-1}{2\pi i} \oint_{z'}^- dw A(z') J'(w) (-\omega')(w) = \frac{-1}{2\pi i} \oint_{z'} dw A(z') J'(w) \omega'(w) \quad (22)$$

where \oint^- denotes an integral with clockwise direction while \oint denotes an integral with counterclockwise direction. We remark that this two-side variation

from (14) and (21) is important for deriving the two KZ equations which are dual to each other. Then similar to (16) we also have

$$\delta_{\omega'} J' = [\omega', J'] - k \partial_{z'} \omega' \quad (23)$$

Then from (22) and (23) we can derive the current algebra and the Kac-Moody algebra for J' which are of the same form of (17) and (19).

6 Dual Knizhnik-Zamolodchikov Equations

Let us first consider (11). From (15) and (11) we have

$$J^a(z)W(w, w') \sim \frac{-t^a W(w, w')}{z - w} \quad (24)$$

Let us define an energy-momentum tensor $T(z)$ by

$$T(z) := \frac{1}{k+g} \sum_a : J^a(z) J^a(z) : \quad (25)$$

where g is the dual Coxeter number. In (25) the symbol $: \dots :$ denotes normal ordering. This is the Sugawara construction of energy-momentum tensor where the appearing of g is from a renormalization of quantum effect by requiring the operator product expansion of T with itself to be of the following form [11] [9] [10]:

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \quad (26)$$

for some constant $c = \frac{kd}{k+g}$ where d denotes the dimension of $SU(2)$. Then we have the following TW operator product:

$$T(z)W(w, w') \sim \frac{\Delta}{(z-w)^2} + \frac{1}{(z-w)} L_{-1}W(w, w') \quad (27)$$

where $L_{-1}W(w, w') = \partial_w W(w, w')$ and

$$\Delta = \frac{\sum_a t^a t^a}{2(k+g)} = \frac{N^2 - 1}{2N(k+N)} \quad (28)$$

where N is for $SU(N)$. From (25) and (27) we have the following equation [9][10]:

$$L_{-1}W(w, w') = \frac{1}{k+g} J_{-1}^a J_0^a W(w, w') \quad (29)$$

Then from (24) we have

$$J_0^a W(w, w') = -t^a W(w, w') \quad (30)$$

By (29) and (30) we have

$$\partial_z W(z, z') = \frac{-1}{k+g} J_{-1}^a t^a W(z, z') \quad (31)$$

From this equation and by the JW operator product (24) we have the following Knizhnik-Zamolodchikov equation [9] [10]:

$$\partial_{z_i} \langle W(z_1, z'_1) \cdots W(z_n, z'_n) \rangle = -\frac{1}{k+g} \sum_{j \neq i}^n \frac{\sum_a t^a \otimes t^a}{z_i - z_j} \langle W(z_1, z'_1) \cdots W(z_n, z'_n) \rangle \quad (32)$$

We remark that in deriving (32) we have used line integral expression of operators with counterclockwise direction [9][10].

It is interesting and important that we also have another Knizhnik-Zamolodchikov equation which will be called the dual equation of (32). The derivation of this dual equation is dual to the above section in that the line integral for this section is with clockwise direction in contrast to counterclockwise direction in the above section and that the operator products and their corresponding variables are with reverse order to that in the above section.

From (12) and (22) we have a WJ' operator product given by

$$W(w, w') J'^a(z') \sim \frac{-W(w, w') t^a}{w' - z'} \quad (33)$$

Similar to the derivation of the KZ equation from (33) we can derive the following Knizhnik-Zamolodchikov equation which is dual to (32):

$$\partial_{z'_i} \langle W(z_1, z'_1) \cdots W(z_n, z'_n) \rangle = -\frac{1}{k+g} \sum_{j \neq i}^n \langle W(z_1, z'_1) \cdots W(z_n, z'_n) \rangle \frac{\sum_a t^a \otimes t^a}{z'_j - z'_i} \quad (34)$$

7 Skein Relation for Jones Polynomial

Following the idea of Witten [1], if we cut a knot we get two pieces of curves crossing (or not crossing) each other once. This gives two primary fields $W(z_1, z_2)$ and $W(z_3, z_4)$ where $W(z_1, z_2)$ corresponds to a piece of curve with end points parametrized by z_1 and z_2 and $W(z_3, z_4)$ corresponds to the other piece of curve with end points parametrized by z_3 and z_4 . Let us write

$$W(z_i, z_j) = W(z_i, z'_i) W^{-1}(z_j, z'_j) \quad (35)$$

for $i = 1, 3$ and $j = 2, 4$ and for some z'_k with $z'_1 = z'_2$ and $z'_3 = z'_4$. These two pieces of curves then correspond to the following four-point correlation function:

$$G(z_1, z_2, z_3, z_4) := \langle W(z_1, z'_1) W^{-1}(z_2, z'_2) W(z_3, z'_3) W^{-1}(z_4, z'_4) \rangle \quad (36)$$

(In the notation $G(z_1, z_2, z_3, z_4)$ we have suppressed the z' variables for simplicity). Then we have [9] [10]:

$$G(z_1, z_2, z_3, z_4) = [(z_1 - z_3)(z_2 - z_4)]^{-2\Delta} G(x) \quad (37)$$

where $\Delta = \frac{N^2-1}{2N(N+k)}$ and $x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$ and from the KZ equation $G(x)$ satisfies the following equation:

$$\frac{dG}{dx} = \left[\frac{1}{x}P + \frac{1}{x-1}Q \right] G \quad (38)$$

where

$$P = -\frac{1}{N(N+k)} \begin{pmatrix} N^2-1 & N \\ 0 & -1 \end{pmatrix}, \quad Q = -\frac{1}{N(N+k)} \begin{pmatrix} -1 & 0 \\ N & N^2-1 \end{pmatrix} \quad (39)$$

This equation has two independent conformal block solutions forming a vector space of dimension 2. Let ψ be a vector in this space and let B denotes the braid operation. Then following Witten [1] we have

$$a\psi - bB\psi + B^2\psi = 0 \quad (40)$$

where $a = \det B$ and $b = \text{Tr} B$. Then following Witten [1] from (40) we can derive the following skein relation for the Jones polynomial:

$$\frac{1}{t}V_{L_-} - tV_{L_+} = (t^{\frac{1}{2}} - \frac{1}{t^{\frac{1}{2}}})V_{L_0} \quad (41)$$

where V_{L_-} , V_{L_+} and V_{L_0} are the Jones polynomials for undercrossing, overcrossing and zero crossing respectively.

8 New Knot Invariants Extending Jones Polynomial

Let us consider again the correlation $G(z_1, z_2, z_3, z_4)$ in (36) which also have the following form:

$$G(z_1, z_2, z_3, z_4) = \langle W(z_1, z_2)W(z_3, z_4) \rangle \quad (42)$$

From it in this section we shall present a method which is different from the above section to derive new knot invariants. These new knot invariants will extend the Jones polynomial and they will be defined by generalizing (6).

We have that G satisfies the KZ equation for the variables z_1, z_3 and satisfies the dual KZ equation for the variables z_2 and z_4 . By solving the KZ equation we have that G is of the form

$$e^{t \log(z_1-z_3)} C_1 \quad (43)$$

where $t := \frac{1}{k+g} \sum_a t^a \otimes t^a$ and C_1 denotes a constant matrix which is independent of the variable $z_1 - z_3$.

Similarly by solving the dual KZ equation we have that G is of the form

$$C_2 e^{-t \log(z_4 - z_2)} \quad (44)$$

where C_2 denotes a constant matrix which is independent of the variable $z_4 - z_2$.

From (43), (44) and we let $C_1 = A e^{-t \log(z_4 - z_2)}$, $C_2 = e^{t \log(z_1 - z_3)} A$ where A is a constant matrix we have that G is given by

$$G(z_1, z_2, z_3, z_4) = e^{t \log(z_1 - z_3)} A e^{-t \log(z_4 - z_2)} \quad (45)$$

Now let $z_2 = z_3$. Then as $z_4 \rightarrow z_1$ we have

$$Tr G(z_1, z_2, z_2, z_1) = Tr e^{i2n\pi t} A =: Tr R^{2n} A \quad n = 0, \pm 1, \pm 2, \dots \quad (46)$$

where $R = e^{i\pi t}$ is the monodromy of the KZ equation [8]. We remark that (46) is a multivalued function. From (46) we have the following relation between the partition function Z and the matrix A :

$$A = IZ \quad (47)$$

where I denotes the identity matrix. Now let C be a closed curve in the complex plane with initial and final end points z_1 . Then the following correlation function

$$Tr \langle W(z_1, z_1) \rangle = Tr \langle W(z_1, z_2) W(z_2, z_1) \rangle \quad (48)$$

which is the definition (6) defined along the curve C , with $W(z_1, z_1) = W(z_1, z_2) W(z_2, z_1)$, can be regarded as a knot invariant of the trivial knot in the three dimensional space whose porjection in the complex plane is the curve C . Indeed, from (42) and (46) we can compute (48) which is given by:

$$Tr \langle W(z_1, z_1) \rangle = Z Tr R^{2n} \quad n = 0, \pm 1, \pm 2, \dots \quad (49)$$

From (49) we see that (48) is independent of the closed curve C which represents the projection of a trivial knot and thus can be regarded as a knot invariant for the trivial knot. In the following let us extend the definition (48) to knot invariants for nontrivial knots.

Since R is the monodromy of the KZ equation, we have a branch cut such that

$$\langle W(z_3, z_2) W(z_1, z_4) \rangle = R \langle W(z_1, z_2) W(z_3, z_4) \rangle \quad (50)$$

where z_1 and z_3 denote two points on a closed curve such that along the direction of the curve the point z_1 is before the point z_3 . From (50) we have

$$W(z_3, z_2) W(z_1, z_4) = R W(z_1, z_2) W(z_2, z_4) \quad (51)$$

Similarly for the dual KZ equation we have

$$W(z_1, z_4)W(z_3, z_2) = W(z_1, z_2)W(z_3, z_4)R^{-1} \quad (52)$$

where z_2 before z_4 . From (51) and (52) we have

$$W(z_3, z_4)W(z_1, z_2) = RW(z_1, z_2)W(z_3, z_4)R^{-1} \quad (53)$$

where z_1 and z_2 denote the end points of a curve which is before a curve with end points z_3 and z_4 . From (53) we see that the algebraic structure of these W matrices is analogous to the quasi-triangular quantum group [8][10]. Now we let $W(z_i, z_j)$ represent a piece of curve with initial end point z_i and final end point z_j . Then we let

$$W(z_1, z_2)W(z_3, z_4) \quad (54)$$

represent two pieces of uncrossing curve. By interchanging z_1 and z_3 we have

$$W(z_3, z)W(z, z_2)W(z_1, z)W(z, z_4) \quad (55)$$

represent the curve specified by $W(z_1, z_2)$ upcrossing the curve specified by $W(z_3, z_4)$ at z . Similarly by interchanging z_2 and z_4 we have

$$W(z_1, z)W(z, z_4)W(z_3, z)W(z, z_2) \quad (56)$$

represent the curve specified by $W(z_1, z_2)$ undercrossing the curve specified by $W(z_3, z_4)$ at z . Now for a closed curve we may cut it into a sum of parts which are formed by two pieces of curve crossing or not crossing each other. Each of these parts is represented by (54), (55) or (56). Then we may define a correlation for a knot represented by this closed curve by the following form:

$$Tr\langle \cdots W(z_3, z)W(z, z_2)W(z_1, z)W(z, z_4) \cdots \rangle \quad (57)$$

where we use (55) as an example to represent the state of the two pieces of curve specified by $W(z_1, z_2)$ and $W(z_3, z_4)$. The \cdots means multiplications of a sequence of parts represented by (54), (55) or (56) according to the state of each part. The order of the sequence in (57) follows the order of the parts given by the direction of the knot. We shall show that (57) is a knot invariant for a given knot. In the following let us consider some examples to illustrate the way to define (57) and to show that (57) is a knot invariant. We shall also derive the three Reidemeister moves for the equivalence of knots.

Let us first consider a knot in Fig. 1. For this knot (57) is given by

$$Tr\langle W(z_2, w)W(w, z_2)W(z_1, w)W(w, z_1) \rangle \quad (58)$$

where the product of W is from the definition (55). In applying (55) we let z_1 be the starting and the final point. We remark that the W matrices must be



Fig.1

put together to follow the definition (55) and they are not separated to follow the direction of the knot. Then we have that (58) is equal to

$$\begin{aligned}
& Tr\langle W(w, z_2)W(z_1, w)W(w, z_1)W(z_2, w) \rangle \\
&= Tr\langle RW(z_1, w)W(w, z_2)R^{-1}RW(z_2, w)W(w, z_1)R^{-1} \rangle \\
&= Tr\langle W(z_1, z_2)W(z_2, z_1) \rangle \\
&= Tr\langle W(z_1, z_1) \rangle
\end{aligned} \tag{59}$$

where we have used (53). We see that (59) is just the knot invariant (48) of a trivial knot. Thus the knot in Fig.1 is with the same knot invariant of a trivial knot and this agrees with the fact that this knot is equivalent to a trivial knot.

Then let us derive the Reidemeister move 1. Consider the diagram in Fig.2. We have that by (55) the definition (57) for this diagram is given by:

$$\begin{aligned}
& Tr\langle W(z_2, w)W(w, z_2)W(z_1, w)W(w, z_3) \rangle \\
&= Tr\langle W(z_2, w)RW(z_1, w)W(w, z_2)R^{-1}W(w, z_3) \rangle \\
&= Tr\langle W(z_2, w)RW(z_1, z_2)R^{-1}W(w, z_3) \rangle \\
&= Tr\langle R^{-1}W(w, z_3)W(z_2, w)RW(z_1, z_2) \rangle \\
&= Tr\langle W(z_2, w)W(w, z_3)W(z_1, z_2) \rangle \\
&= Tr\langle W(z_2, z_3)W(z_1, z_2) \rangle \\
&= Tr\langle W(z_1, z_3) \rangle
\end{aligned} \tag{60}$$

where $W(z_1, z_3)$ represent a piece of curve with initial end point z_1 and final end point z_3 which has no crossing. When Fig.2 is a part of a knot we can also derive a result similar to (60) which is for the Reidemeister move 1. This shows that the Reidemeister move 1 holds. Then let us derive Reidemeister move 2.

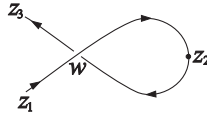


Fig.2

By (55) we have that the definition (57) for the two pieces of curve in Fig.3a is given by

$$\begin{aligned}
& Tr\langle W(z_5, w_1)W(w_1, z_2)W(z_1, w_1)W(w_1, z_6) \cdot \\
& W(z_4, w_2)W(w_2, z_3)W(z_2, w_2)W(w_2, z_5) \rangle
\end{aligned} \tag{61}$$

where the two products of W separated by the \cdot are for the two crossings in

Fig.3a. We have that (61) is equal to

$$\begin{aligned}
& Tr\langle W(z_4, w_2)W(w_2, z_3)W(z_2, w_2)W(w_2, z_5) \cdot \\
& W(z_5, w_1)W(w_1, z_2)W(z_1, w_1)W(w_1, z_6) \rangle \\
& = Tr\langle W(z_1, z_3)W(z_4, z_6) \rangle
\end{aligned} \tag{62}$$

where we have repeatedly used (53). This shows that the diagram in Fig.3a is equivalent to two uncrossing curves. When Fig.3a is a part of a knot we can also derive a result similar to (62) for the Reidemeister move 2. This shows that the Reidemeister move 2 holds. As an illustration let us consider the knot

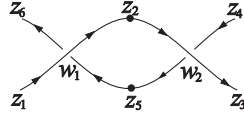


Fig.3a

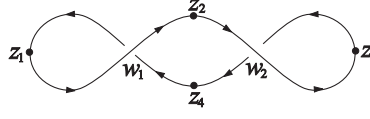


Fig.3b

in Fig.3b which is related to the Reidemeister move 2. By (55) we have that the definition (57) for this knot is given by

$$\begin{aligned}
& Tr\langle W(z_3, w_2)W(w_2, z_3)W(z_2, w_2)W(w_2, z_4) \cdot \\
& W(z_4, w_1)W(w_1, z_2)W(z_1, w_1)W(w_1, z_1) \rangle \\
& = Tr\langle RW(z_2, w_2)W(w_2, z_3)W(z_3, w_2)W(w_2, z_4) \cdot \\
& W(z_4, w_1)W(w_1, z_1)W(z_1, w_1)W(w_1, z_2)R^{-1} \rangle \\
& = Tr\langle W(z_2, w_2)W(w_2, z_3)W(z_3, w_2)W(w_2, z_4) \cdot \\
& W(z_4, w_1)W(w_1, z_1)W(z_1, w_1)W(w_1, z_2) \rangle \\
& = Tr\langle W(z_2, z_2) \rangle
\end{aligned} \tag{63}$$

where we let the curve be with z_2 as the initial and final end point and we have used (51) and (52). This shows that the knot in Fig.3b is with the same knot invariant of a trivial knot. This agrees with the fact that this knot is equivalent to the trivial knot. Similar to the above derivations we can derive the Reidemeister move 3.

Let us then consider a trefoil knot in Fig.4a. By (55) and similar to the above examples we have that the definition (57) for this knot is given by:

$$\begin{aligned}
& Tr\langle W(z_4, w_1)W(w_1, z_2)W(z_1, w_1)W(w_1, z_5) \cdot W(z_2, w_2)W(w_2, z_6) \\
& W(z_5, w_2)W(w_2, z_3) \cdot W(z_6, w_3)W(w_3, z_4)W(z_3, w_3)W(w_3, z_1) \rangle \\
& = Tr\langle W(z_6, z_1)W(z_3, z_6)W(z_1, z_3) \rangle
\end{aligned} \tag{64}$$

where we have repeatedly used (53). Then we have that (64) is equal to:

$$\begin{aligned}
& Tr\langle W(z_6, w_3)W(w_3, z_1)W(z_3, w_3)W(w_3, z_6)W(z_1, z_3) \rangle \\
&= Tr\langle RW(z_3, w_3)W(w_3, z_1)W(z_6, w_3)W(w_3, z_6)W(z_1, z_3) \rangle \\
&= Tr\langle RW(z_3, w_3)RW(z_6, w_3)W(w_3, z_1)R^{-1}W(w_3, z_6)W(z_1, z_3) \rangle \\
&= Tr\langle W(z_3, w_3)RW(z_6, z_1)R^{-1}W(w_3, z_6)W(z_1, z_3)R \rangle \\
&= Tr\langle W(z_3, w_3)RW(z_6, z_3)W(w_3, z_6) \rangle \\
&= Tr\langle W(w_3, z_6)W(z_3, w_3)RW(z_6, z_3) \rangle \\
&= Tr\langle RW(z_3, w_3)W(w_3, z_6)W(z_6, z_3) \rangle \\
&= Tr\langle RW(z_3, z_3) \rangle
\end{aligned} \tag{65}$$

where we have used (51) and (53). We see that (65) is a knot invariant for the trefoil knot in Fig.4a. Similarly for the trefoil knot in Fig. 4b which is the

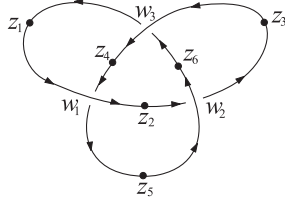


Fig.4a

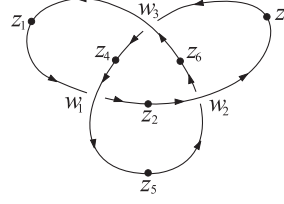


Fig.4b

mirror image of the trefoil knot in Fig.4a we have that the definition (57) for this knot is equal to $Tr\langle W(z_5, z_5)R^{-1} \rangle$ which is a knot invariant for the trefoil knot in Fig.4b. We notice that this knot invariant is different from (65). This shows that these two trefoil knots are not topologically equivalent.

With the above new knot invariants we can now give a classification of knots. Let C_1 and C_2 be two knots. Then since we can derive the Reidemeister moves by using (51), (52) and (53), we have that C_1 and C_2 are topologically equivalent if and only if the W -product and the correlation (57) for C_1 can be transformed to the W -product and the correlation (57) for C_2 by using (51), (52) and (53). Thus correlations (57) are knot invariants which can completely classify knots. From the above examples we see that knots can be classified by the number of product of R and R^{-1} matrices. More calculations and examples of the above knot invariants will be given elsewhere.

References

- [1] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. **121**, 351 (1989)
- [2] S.K. Ng, Information and system modelling, Math. and Comp. Modelling. **23,5**, 1 (1996)

- [3] S.K. Ng, New gauge model of QED and unification of statistical and quantum physics, In *Statistical Models, Yang-Baxter Equation and Related Topics*, eds. Ge M.L. and Wu F.Y., pp. 262-276 (World Scientific 1996).
- [4] P.A.M. Dirac, *Directions in Physics*, (John Wiley, 1978).
- [5] C. Itzykson and J-B. Zuber, *Quantum Field Theory*, (McGraw-Hill Inc., 1980).
- [6] L. Kauffman, *Knots and Physics*, (World Scientific 1993).
- [7] J. Baez and J. Muniain, *Gauge Fields, Knots and Gravity*, (World Scientific 1994).
- [8] V. Chari and A. Pressley, *A Guide to Quantum Groups*, (Cambridge University Press 1994).
- [9] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, (Springer-Verlag 1997).
- [10] J. Fuchs, *Affine Lie Algebras and Quantum Groups*, (Cambridge University Press 1992).
- [11] V.G. Knizhnik and A.B. Zamolodchikov, Current algebra and Wess-Zumino model in two dimensions, Nucl. Phys. B **247**, 83 (1984).